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Pontrjagin forms, Chern Simons classes, Codazzi transformations, and affine hypersurfaces

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Abstract

We show that the primary and secondary characteristic classes vanish in the context of affine differential geometry. This gives rise to obstructions to realizing a conformal class of metrics on a manifold either as the first or as the second fundamental form of an affine immersion. © 1998 Elsevier Science B.V.

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1. Introduction

The primary and secondary characteristic forms are the focus of our study in this paper. Consequently, it is worth motivating their study; they appear in many contexts. Let M be a compact m-dimensional manifold with smooth boundary ∂M. Let h be a Riemannian metric on M and let R be the curvature tensor of the Levi-Civita connection ${}^h\nabla$. Let indices i, j, k, and l range from 1 to m and index a local orthonormal frame $\{e_i\}$ for the tangent bundle. At a point of the boundary of M, we assume that e_m is the inward unit normal and let indices a, b, c range from 1 to $m-1$. Let $L_{ab} := ({}^h\nabla_{e_a} e_b, e_m)$ be the second fundamental form on the boundary. We adopt the Einstein convention and sum over repeated indices. Suppose

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that M is a compact four dimensional orientable manifold with smooth boundary ∂M. Let $\mathrm{d}x$ and $\mathrm{d}y$ be the volume elements on M and on ∂M. The Chern–Gauss–Bonnet formula [8] and the Atiyah–Patodi–Singer index formula [2] yield formulas for the Euler–Poincare characteristic $\chi(M)$ and the signature $\mathrm{Sign}(M)$:

$$\begin{aligned}\chi(M) &= \int_M E_4({}^h\nabla)\,\mathrm{d}x - \int_{\partial M} TE_4({}^h\nabla, L)\,\mathrm{d}y,\\ \mathrm{Sign}(M) &= \frac{1}{3}\int_M P_1({}^h\nabla) - \frac{1}{3}\int_{\partial M} TP_1({}^h\nabla, L) - \eta(\partial M).\end{aligned} \tag{1.1}$$

In these formulas we have

$$\begin{aligned}E_4({}^h\nabla) :&= \frac{1}{32\pi^2}(R_{ijji}R_{kllk} - 4R_{ijjk}R_{illk} + R_{ijkl}R_{ijkl}),\\ P_1({}^h\nabla) :&= -\frac{1}{32\pi^2}R_{ijk_1k_2}R_{jik_3k_4}e^{k_1}\wedge e^{k_2}\wedge e^{k_3}\wedge e^{k_4},\\ TE_4({}^h\nabla) :&= -\frac{1}{24\pi^2}(3R_{ijji}L_{aa} + 6R_{amam}L_{bb} + 6R_{acbc}L_{ab}\\ &\quad +2L_{aa}L_{bb}L_{cc} - 6L_{ab}L_{ab}L_{cc} + 4L_{ab}L_{bc}L_{ca}),\\ TP_1(L, {}^h\nabla) :&= -\frac{1}{16\pi^2}L_{ab}R_{4acd}e^b\wedge e^c\wedge e^d,\end{aligned} \tag{1.2}$$

The interior integrands E_4 and P_1 are *primary characteristic forms*; the boundary integrand TE_4 and TP_1 are *secondary characteristic forms*. The invariant $\eta(\partial M)$ is intrinsic to ∂M and is a global spectral invariant of ∂M; we will not be concerned with this invariant here. There are suitable generalizations of these formulas to higher dimensions, see [16] for details. These formulas play a crucial role in the study of gravitational instantons; see [12] for a more complete bibliography. The primary and secondary characteristic forms are crucial in index theory for manifolds with boundary. In these formulas we are dealing with differential forms and not with cohomology classes; the index theorem involves geometric quantities and not topological ones in this setting.

In addition to their use in the index theorem, the secondary characteristic forms are important in the study of three-dimensional geometry. They give rise to invariants of knots, see for example, [3,13,22]. They give rise to invariants of hyperbolic manifolds [23]. Horava [21] uses them to study orbifolds. Finally, they are important in mathematical physics. Chae and Kim [7] use them to study the Maxwell–Chern–Simons–Higgs system; Haller and Lombridas [18] use them to study quantum electrodynamics in $2 + 1$ dimensions; Pashaev [24] and Yang [33] use them to study gauge theory. We refer to [15] for a formulation of Chern–Simons theory as a standard classical field theory.

The primary and secondary characteristic classes form one theme for our present study. The second theme comes from affine differential geometry and from the study of Codazzi transformations. We quote from [6] "Relations between conformal and projective structures are of particular interest in both mathematics and in mathematical physics. Weyl [31] attempted a unification of gravitation and electromagnetism in a model of space–time geometry combining both structures. His particular approach failed for physical reasons but his model is still studied in mathematics (see, for example [14,19,25]) and in mathematical physics (see, for example [20])". In addition to applications in mathematical physics, the Codazzi tensor arises naturally in affine differential geometry, see for example [28]. We also refer to [4,11,27,30] for related work on the Codazzi tensor.

Here is a brief outline to the paper. In Section 2, we review the construction of the primary characteristic forms $Q(\nabla)$ where Q is an invariant polynomial and where ∇ is a connection on TM; we refer to [12,17] for further details concerning this material. Let ${}^h\nabla$ be the Levi-Civita connection of a semi-Riemannian metric h on M. Note that if h and $\tilde{h}$ are conformally equivalent, then $Q({}^h\nabla) = Q({}^{\tilde{h}}\nabla)$; the characteristic forms are conformal invariants. Two torsion free connections ∇ and $\tilde{\nabla}$ are said to be projectively equivalent if their unparametrized geodesics agree. We will show that if ∇ and $\tilde{\nabla}$ are projectively equivalent, then $Q(\nabla) = Q(\tilde{\nabla})$; the characteristic forms are also projective invariants.

In Section 3, we give a brief introduction to affine differential geometry; we refer to [5,29] for further information concerning this material. If x is a non-degenerate embedding of a manifold M as a hypersurface in affine space, we let (x, X, y) be a relative normalization. This defines a triple $(\nabla, h, {}^*\nabla)$ where h is a semi-Riemannian metric on M and where ∇ and ${}^*\nabla$ are torsion free connections on the tangent bundle. If Q is an invariant polynomial, we will show that $Q(\nabla) = 0$, that $Q({}^h\nabla) = 0$, and that $Q({}^*\nabla) = 0$, see Theorem 3.2.

In Section 4, we review the construction of the absolute and relative secondary characteristic classes. We refer to [9,10,12] for further details concerning this material. The relative secondary characteristic forms arise from considering pairs of connections and the absolute secondary characteristic forms arise from the transgression and are defined on the principal bundle. In Section 5, we show that the secondary characteristic forms of the connections ∇, ${}^*\nabla$, and ${}^h\nabla$ vanish, see Theorem 5.2. In Section 6, we apply these results to three-dimensional affine differential geometry to construct obstructions to realizing the conformal class of a Riemannian metric as the second fundamental form of an embedding, see Theorem 6.1; this generalizes work of Chern–Simons [10].

2. Pontrjagin forms and second characteristic classes

We shall restrict our attention to the tangent bundle TM henceforth; let ∇ be an arbitrary connection on TM. The curvature R of ∇ is given by

$$R(u, v) := \nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{(u,v)},$$

where u and v are vector fields on M. If $\{e_i\}$ is a local frame for TM, then $R = R_{ijk}{}^l$ where $R(e_i, e_j)e_k = R_{ijk}{}^l e_l$; we adopt the Einstein convenion and sum over repeated indices. We shall let

$$\mathcal{R} = \mathcal{R}_k^l := \tfrac{1}{2}R_{ijk}{}^l e^i \wedge e^j$$

be the associated 2-form valued endomorphism. As we are not assuming that a metric is given, we do not restrict to orthonormal frames. Thus the structure group is the full general linear group $\mathrm{GL}(m; \mathbb{R})$ and not the orthogonal group $\mathrm{O}(m)$.

Definition 2.1. Let $\mathfrak{gl}(m, \mathbb{R})$ be the Lie algebra of $\mathrm{GL}(m; \mathbb{R})$; this is the Lie algebra of real $m \times m$ matrices. If Q is a map from $\mathfrak{gl}(m; \mathbb{R})$ to $\mathbb{C}$, we say that Q is *invariant* if $Q(gAg^{-1}) = Q(A)$ for all $A \in \mathfrak{gl}(m; \mathbb{R})$ and for all $g \in \mathrm{GL}(m; \mathbb{R})$. Let $\mathcal{Q}$ be the ring of invariant polynomials. We decompose $\mathcal{Q} = \oplus \mathcal{Q}_\nu$ as a graded ring where $\mathcal{Q}_\nu$ is the subspace of invariant polynomials which are homogeneous of degree ν. Let

$$Ch(A) := \sum_\nu Ch_\nu \quad \text{for } Ch_\nu(A) := \mathrm{Tr}\left\{\left(\frac{\sqrt{-1}}{2\pi}A\right)^\nu\right\},$$

and

$$C(A) := \det\left(I + \frac{\sqrt{-1}}{2\pi}A\right) = 1 + C_1(A) + \cdots + C_m(A)$$

define the *Chern character* and total *Chern polynomial*; $Ch_\nu \in \mathcal{Q}_\nu$ and $C_\nu \in \mathcal{Q}_\nu$. The Chern characters and the Chern polynomials generate the characteristic ring:

$$\mathcal{Q} = \mathbb{C}[C_1, \ldots, C_m] \quad \text{and} \quad \mathcal{Q} = \mathbb{C}[Ch_1, \ldots, Ch_m].$$

Let $Q \in \mathcal{Q}_\nu$. We polarize Q to define a multi-linear form $Q(A_1, \ldots, A_\nu)$ so that

$$Q(A) = Q(A, \ldots, A) \quad \text{and} \quad Q(A_1, \ldots, A_\nu) = Q(gA_1g^{-1}, \ldots, gA_\nu g^{-1}).$$

If $Q \in \mathcal{Q}_\nu$, we define

$$Q(\nabla) := Q(\mathcal{R}, \ldots, \mathcal{R}) \in C^\infty(\Lambda^{2\nu}M)$$

by substitution; the value is independent of the frame chosen and associates a closed differential form of degree 2ν to any connection ∇ on TM. The corresponding cohomology class $(Q(\nabla)) \in H^{2\nu}(M; \mathbb{C})$ is independent of the connection ∇ chosen; see Eq. (4.1). These are the characteristic forms and classes.

Let $A \in \mathfrak{o}(m)$ be a skew-symmetric matrix. Then $C_{2\nu+1}(A) = 0$ and we define $P_\nu(A) = (-1)^\nu C_{2\nu}(A)$; $P = \sum_\nu P_\nu(A)$ is the total *Pontrjagin polynomial.* The $\{P_\nu\}$ for $2\nu \le m$ generate the characteristic ring of the orthogonal group $\mathrm{O}(m)$. We can always choose a Riemannian metric g for M and use the associated Levi-Civita connection ${}^g\nabla$ to compute the characteristic classes of the tangent bundle. This reduces the structure group to $\mathrm{O}(m)$ and shows that only the Pontrjagin classes are relevant in the study of the primary characteristic classes of TM. From the point of view of cohomology, the connection plays an inessential role; however, as noted in Section 1, in many geometrical applications, one must work with differential forms and not cohomology classes; it is the differential form P_1 and not the characteristic class (P_1) which plays a crucial role in Eqs. (1.1) and (1.2).

Definition 2.2. Let $\mathcal{G} := C_+^\infty(M)$ be the space of smooth positive functions on M; this is a group under pointwise multiplication and will be our gauge group. The associated Lie algebra is $C^\infty(M)$ and the map $\alpha \mapsto e^\alpha$ provides the usual exponential correspondence. Since $\mathcal{G}$ is Abelian, the Lie bracket is trivial. Let ${}^h\nabla$ be Levi-Civita connection associated to a semi-Riemannian metric h. The gauge group $\mathcal{G}$ acts on the set of semi-Riemannian metrics by conformal rescaling;

$$\beta(h) := \beta h \quad \text{for } g \in \mathcal{G}. \tag{2.1}$$

If ∇ is a torsion free connection and if $f \in C^\infty(M)$, then the *Hessian* $H_\nabla(f)$ is a 2-tensor given by

$$H_\nabla(f)(u, v) := u(v(f)) - (\nabla_u v)(f).$$

Since ∇ is torsion free, we have $\nabla_u v - \nabla_v u = (u, v)$ and thus $H_\nabla(f)$ is a symmetric tensor. If we use ';' to denote multiple covariant differentiation, then the components of $H_\nabla(f)$ are given by $f_{;ij}$ and we have $f_{;ij} = f_{;ji}$.

We say that two torsion free connections ∇ and $\tilde{\nabla}$ are *projectively equivalent* if their unparametrized geodesics coincide. Equivalently, this means that there is a smooth closed 1-form θ so that

$$(\tilde{\nabla}_u - \nabla_u)v = \theta(u)v + \theta(v)u;$$

we refer to [26] for details. Locally, we can always choose a primitive for θ and express $\theta = d \ln \beta$. We define an action of the gauge group $\mathcal{G}$ on the set of torsion free connections by defining

$$\beta(\nabla)_u v := \nabla_u v + u(\ln \beta)v + v(\ln \beta)u. \tag{2.2}$$

Conformally equivalent metrics and projectively equivalent torsion free connections have the same characteristic forms:

Theorem 2.3. *Let $Q \in \mathcal{Q}_\nu$ and let $\beta \in \mathcal{G}$.*
(1) *Let ${}^h\nabla$ be the Levi-Civita connection of a semi-Riemannian metric. Then $Q({}^h\nabla) = Q({}^{\beta(h)}\nabla)$.*
(2) *Let ∇ be a torsion free connection. Then $Q(\beta(\nabla)) = Q(\nabla)$.*

Proof. Although assertion (1) is well-known, we give a somewhat non-standard proof adapting an argument of Atiyah et al. [1] to motivate the proof we shall give for assertion (2). Let ${}^hR_{ijk}{}^l$ be the components of the curvature tensor of the Levi-Civita connection ${}^h\nabla$. We use Weyl's theorem [32] on the invariants of the orthogonal group to see that Q can be expressed in terms of traces. This means that we must alternate 2ν indices and contract the remaining indices in pairs. For example,

$$P_1(R) = -\frac{1}{32\pi^2} R_{ijk_1k_2} R_{jik_3k_4} e^{k_1} \wedge e^{k_2} \wedge e^{k_3} \wedge e^{k_4}.$$

The indices i and j are contracted; the indices k_ν are alternated. We refer to [17] for a more detailed discussion. Let $\alpha := \ln \beta$. We define a 1-parameter subgroup $\beta(t) := e^{t\alpha}$ of $C^\infty_+(M)$. Let ${}^h\nabla(t)$ be the Levi-Civita connections of the metrics $h(t) := \beta(t)h$. We linearize the variation and define

$$\delta Q(\alpha, {}^h\nabla) := \mathrm{d}_t Q({}^h\nabla(t))|_{t=0}.$$

To complete the proof of the first assertion, we must show $\delta Q = 0$. If ∂_i is a local coordinate frame for TM, let $\Gamma_{ij}{}^k := \frac{1}{2}h^{kl}(\partial_j h_{il} + \partial_i h_{jl} - \partial_l h_{ij})$ be the Christoffel symbols. We have

$$\partial_t R_{ijk}{}^l(t)|_{t=0} = \tfrac{1}{2}h^{ln}(h_{jn}\alpha_{;ik} + h_{ik}\alpha_{;jn} - h_{in}\alpha_{;jk} - h_{jk}\alpha_{;in}),$$
$$\delta Q(\alpha, {}^h\nabla) = \sum_{ij}\alpha_{;ij}Q_{ij}({}^h\nabla).$$

The polynomials Q_{ij} are homogeneous of degree $\nu - 1$ in the components of R. We apply Weyl's theorem [32] on the invariants of the orthogonal group. There are a total of $2 + 4(\nu - 1) = 4\nu - 2$ indices present in a typical monomial of δQ. We must alternate 2ν of these indices and contract $2\nu - 2$ of these indices in pairs to form invariant expressions. Since $\alpha_{;ij}$ is a symmetric 2-tensor, we cannot alternate both of the indices which appear in $\alpha_{;ij}$. Thus we must alternate at least $2\nu - 1$ of the $4\nu - 4$ indices appearing in the $\nu - 1$ R-variables. Thus at least three indices are alternated in some R variable. The Bianchi identities $R_{ijkl} + R_{jkil} + R_{kijl} = 0$ and the other curvature symmetries then imply that this alternation vanishes. This completes the proof of the first assertion.

The proof of the second assertion is similar. Let $\nabla(t)$ be the 1-parameter family of projectively equivalent connections defined by the action of $\beta(t)$ on the set of torsion free connections given in Eq. (2.2). We use [5, Lemma 2.1] to see

$$R_t(u, v)w = R(u, v)w + e^{t\alpha}H_\nabla(e^{-t\alpha})(v, w)u - e^{t\alpha}H_\nabla(e^{-t\alpha})(u, w)v,$$
$$\partial_t R_t(u, v)w|_{t=0} = H_\nabla(\alpha)(u, w)v - H_\nabla(\alpha)(v, w)u,$$
$$\partial_t R_{ijk}{}^l|_{t=0} = \alpha_{;ik}\delta_j^l - \alpha_{;jk}\delta_i^l,$$
$$\delta Q(\alpha, \nabla) := \mathrm{d}_t Q(\nabla(t))|_{t=0} = \sum_{ij}\alpha_{;ij}Q_{ij}(R).$$

The coefficients $Q_{ij}(R)$ are multi-linear expressions which are homogeneous of degree $\nu - 1$ in R. Since ∇ is torsion free, $\alpha_{;ij} = \alpha_{;ji}$.

The natural structure group in this setting is $\mathrm{GL}(m; \mathbb{R})$, not the orthogonal group. Thus the distinction between upper and lower indices is crucial. We have $\nu - 1$ upper indices and $3\nu - 1$ lower indices which are free. We must contract $\nu - 1$ upper indices against $\nu - 1$ lower indices and alternate the remaining 2ν lower indices. Since $\alpha_{;ij}$ is symmetric, we cannot alternate two indices in $\alpha_{;ij}$. Thus we must alternate at least $2\nu - 1$ lower indices in the $\nu - 1$ R variables. Again, a counting argument shows that we must alternate three lower indices in some R variable. Since ∇ is torsion free, the Bianchi identity holds for the curvature R and this alternation vanishes. □

3. Affine differential geometry

We shall begin this section with a brief introduction to affine and Codazzi geometry. Let $\mathcal{A}$ be a real affine space which is modeled on a vector space V of dimension $m + 1$. If $a \in \mathcal{A}$, we identify the tangent space $T_a\mathcal{A}$ with V and the cotangent space $T_a^*\mathcal{A}$ with the dual vector space V^*. Let $\langle\cdot,\cdot\rangle : V^* \otimes V \to \mathbb{R}$ be the natural pairing between V^* and V.

Let $x : M \to \mathcal{A}$ be a smooth immersion of M into $\mathcal{A}$ as a non-singular hypersurface. The conormal space at a point P of M is defined by

$$C(M)_P := \{X \in V^* : \langle X,\, \mathrm{d}x(u)\rangle = 0 \;\forall\, v \in T_P M\}.$$

By passing to a suitable double cover of M if necessary, we may assume that the conormal bundle $C(M)$ is trivial and choose a non-vanishing conormal vector field X. We say that the immersion x is *regular* if and only if there exists X such that $\mathrm{Rank}(X,\, \mathrm{d}X) = m + 1$ for all points P of M; we impose this condition henceforth. Define $y = y(X) : M \to V$ by the conditions

$$\langle X, y\rangle = 1 \quad\text{and}\quad \langle \mathrm{d}X, y\rangle = 0.$$

The triple (x, X, y) is called a hypersurface with relative normalization; we note that y need not be an immersion. We define a transitive action of the gauge group $\mathcal{G}$ on the set of relative normalizations by rescaling X; we set

$$\beta(x, X, y) := (x, \beta X, \beta^{-1} \cdot (y + \mathrm{d}X(\mathrm{grad}_h \ln \beta))). \tag{3.1}$$

Let ${}^{\mathcal{A}}\nabla$ be the flat affine connection on $\mathcal{A}$. The relative structure equations given below contain the fundamental geometric quantities of hypersurface theory: two connections ∇, ${}^*\nabla$, the relative shape or Weingarten operator S, and two symmetric forms h and $\hat{S}$. We have

$$\begin{aligned} {}^{\mathcal{A}}\nabla_v y &= \mathrm{d}y(v) = -\,\mathrm{d}x(S(v)),\\ {}^{\mathcal{A}}\nabla_u\, \mathrm{d}x(v) &= \mathrm{d}x(\nabla_u v) + h(u, v)y,\\ \nabla_u^{\mathcal{A}}\, \mathrm{d}X(v) &= \mathrm{d}X({}^*\nabla_u v) - \tilde{S}(u, v)X. \end{aligned} \tag{3.2}$$

The first equation is called the Weingarten equation, and the second two are called the Gauss equations. The tensor h is called the Blaschke metric; we assume that it is non-degenerate. Note that $h(u, S(v)) = \tilde{S}(u, v)$.

Definition 3.1. We say that a torsion free connection ${}^*\nabla$ and a semi-Riemannian metric h satisfy the *Codazzi equations* or are *Codazzi compatible* if

$$({}^*\nabla_u h)(v, w) = ({}^*\nabla_v h)(u, w).$$

Note that if $(h, {}^*\nabla)$ satisfies the Codazzi equation, then $(\beta(h), \beta({}^*\nabla))$ also satisfies the Codazzi equation, so Codazzi compatibility is preserved by the action of the gauge group $\mathcal{G}$ given in Eqs. (2.1) and (2.2). We say that $(\nabla, h, {}^*\nabla)$ is a *conjugate triple* if ∇ and ${}^*\nabla$ are torsion free connections, if h is a semi-Riemannian metric on M, if $(h, {}^*\nabla)$ satisfies the Codazzi equation, and if we have the metric duality identity:

$$uh(v, w) = h(\nabla_u v, w) + h(v, {}^*\nabla_u w).$$

Note that in this setting (h, ∇) is Codazzi compatible as well. Given a torsion free connection ∇ and a semi-Riemannian metric h, we define the 3-tensor

$$C(h, \nabla)_{ijk} := h(({}^h\nabla_{e_i} - \nabla_{e_i})e_j, e_k),$$

where $\{e_i\}$ is a local orthonormal frame field. Suppose that $(\nabla, h, {}^*\nabla)$ satisfies the duality equation given above. Then $(\nabla, h, {}^*\nabla)$ is a conjugate triple if and only if

$$C_{ijk} = C_{jik} = C_{ikj} \quad \text{and} \quad C_{iij;k} = C_{iik;j}.$$

We define an action of the gauge group $\mathcal{G}$ on the set of conjugate triples by setting

$$a(\beta(\nabla, h, {}^*\nabla)) = (a(\beta, h)\nabla, \beta(h), \beta({}^*\nabla)),$$

where

$$\begin{aligned}
&(a(\beta, h)\nabla)_u v := \nabla_u v - h(u, v)\,\mathrm{grad}_h \ln\beta,\\
&\beta(h) := \beta h,\\
&\beta({}^*\nabla)_u v := {}^*\nabla_u v + u(\ln\beta)v + v(\ln\beta)u.
\end{aligned} \tag{3.3}$$

If (x, X, y) is a relative normalization of a non-degenerate hypersurface, then $(\nabla, h, {}^*\nabla)$ is a conjugate triple. The action of the gauge group $\mathcal{G}$ on the set of relative normalizations given in Eq. (3.1) is compatible with the action of $\mathcal{G}$ on the set of associated conjugate triples given in Eq. (3.3). The following is one of the main results of this paper:

Theorem 3.2. *Let $Q \in \mathcal{Q}_\nu$ and let $(\nabla, h, {}^*\nabla)$ be the conjugate triple defined by a relative normalization (x, X, y). Then $Q(\nabla) = 0$, $Q({}^h\nabla) = 0$, and $Q({}^*\nabla) = 0$.*

Proof. We use the metric to raise and lower indices. We have by [29, pp. 72 and 78] that

$$\begin{aligned}
&\hat{S}(u, v) = h(S(u), v) = h(u, S(v)),\\
&R(\nabla)_{rsi}{}^l = h_{si}S_r^l - h_{ri}S_s^l,\\
&R({}^*\nabla)_{rsi}{}^l = S_{si}\delta_r^l - S_{ri}\delta_s^l,\\
&R({}^h\nabla)_{rsi}{}^l = C_{ri}{}^a C_{as}{}^l - C_{si}{}^a C_{ar}{}^l + \tfrac{1}{2}\{\hat{S}_{si}\delta_r^l - \hat{S}_{ri}\delta_s^l + h_{si}S_r^l - h_{ri}S_s^l\}.
\end{aligned} \tag{3.4}$$

Thus we may express $Q(\nabla) = Q_1(h, \hat{S})$ and $Q({}^*\nabla) = Q_2(h, \hat{S})$ where Q_i is homogeneous of degree ν in the symmetric tensor $\hat{S}$. We have 2ν indices coming from $\hat{S}$ at our disposal in each monomial of Q_i. Weyl's theorem [32] shows that to construct a 2ν form, we must alternate all these indices. This yields 0 as $\hat{S}$ is a symmetric tensor. Consequently $Q(\nabla) = 0$ and $Q({}^*\nabla) = 0$.

The situation with ${}^h\nabla$ is a bit more complicated. Express $Q({}^h\nabla) = Q_3(C, \hat{S})$. We define the degree of C to be 1 and the degree of $\hat{S}$ to be 2; this counts the number of derivatives which appear. We use Eq. (3.4) to see that $R({}^h\nabla)$ is homogeneous of degree 2 and so Q_3 is homogeneous of degree 2ν. There are three indices in each C variable and two indices in each $\hat{S}$ variable. If a monomial contains c of the C variables and s of the $\hat{S}$ variables, then $c + 2s = 2\nu$ and there are a total of $3c + 2s$ variables. We must alternate 2ν indices and contract the remaining $3c + 2s - 2\nu = 2c$ indices in pairs. Since C is symmetric, we can only alternate only one index in each C variable. Thus we must alternate all the indices which appear in the S variables. Since S is symmetric, this yields 0 if $s > 0$.

This argument shows that the monomials which contain S are trivial and hence $Q_3 = Q_3(C)$. We have 2ν of the C variables; in each variable, we alternate one index. The remaining indices are contracted in pairs. Thus we can break up Q_3 as the product of cycles of length L which have the form:

$$C_{i_1,i_2,j_1} C_{i_2,i_3,j_2} \cdots C_{i_L,i_1,j_L} e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_L}. \tag{3.5}$$

Since S does not appear in Q_3, we may replace the curvature endomorphism which we defined to be $R^l_k := R_{ijk}{}^l e^i \wedge e^j$ by a new endomorphism

$$D^l_k := (C_{rk}{}^j C_{sj}{}^l - C_{sk}{}^j C_{rj}{}^l) e^r \wedge e^s = 2C_{rk}{}^j C_{sj}{}^l e^r \wedge e^s.$$

This is quadratic in C. We form monomials of Q_3 by contracting indices of D in pairs; this shows that all the cycles which appear in Eq. (3.5) have even length L. We perform a cyclic permutation

$$i_1 \mapsto i_2 \mapsto \cdots \mapsto i_L \mapsto i_1 \text{ and } j_1 \mapsto j_2 \mapsto \cdots \mapsto j_L \mapsto j_1.$$

Since C is totally symmetric and since L is even, the cycle of Eq. (3.5) changes sign under this permutation. This shows $Q_3 = 0$. □

4. Secondary characteristic forms

4.1. Relative characteristic forms

The space of all connections is an affine space; the space of torsion free connections is an affine subspace. If ∇_i are connections on TM, let $\nabla_t := t\nabla_1 + (1-t)\nabla_0$. Let $\psi := \nabla_1 - \nabla_0$; ψ is an invariantly defined 1 form valued endomorphism. Let $R(t)$ be the associated curvature. Let $Q \in \mathcal{Q}_\nu$. Let

$$TQ(\nabla_1, \nabla_0) := \nu \int_0^1 Q(\psi, R(t), \ldots, R(t))\, \mathrm{d}t,$$

$$\mathrm{d}TQ(\nabla_1, \nabla_0) = Q(\nabla_1) - Q(\nabla_0). \tag{4.1}$$

This shows that $[Q(\nabla_1)] = [Q(\nabla_2)]$ in de Rham cohomology as discussed in Section 1. Note that we have

$$TQ(\nabla_0, \nabla_1) + TQ(\nabla_1, \nabla_2) = TQ(\nabla_0, \nabla_2) + \text{exact}.$$

Suppose that M is a four-dimensional Riemannian manifold with smooth non-empty boundary ∂M. Choose a Riemannian metric h on M. Let $x = (y, t)$ be local coordinates for M near ∂M so the curves $t \mapsto (y, t)$ are unit speed geodesics perpendicular to the boundary. This identifies a neighborhood of ∂M in M with a collared neighborhood $\mathcal{K} := \partial M \times [0, \epsilon)$ for some $\epsilon > 0$. Let h_0 be the associated product metric. Let ∇_1 be the Levi-Civita connection of h and ∇_0 the Levi-Civita connection of h_0. Then $TP_1(\nabla_1, \nabla_0)$ is given by Eq. (1.2); see [12] for details.

4.2. Absolute secondary characteristic forms

Let $\pi : \mathcal{P} \to M$ be the principal frame bundle for TM; a local section e to $\mathcal{P}$ is a frame $e = \{e_i\}$ for TM. Let g be the natural inclusion of $\mathrm{GL}(m, \mathbb{R})$ in the Lie algebra $\mathfrak{gl}(m; \mathbb{R})$ of $m \times m$ real matrices. The Maurer–Cartan form $\mathrm{d}g\, g^{-1}$ on $\mathrm{GL}(m; \mathbb{R})$ is a $\mathfrak{gl}(m; \mathbb{R})$ valued 1-form on $\mathrm{GL}(m; \mathbb{R})$ which is invariant under right multiplication. Let ∇ be a connection on TM. Fix a local frame field e for TM; this is often called a *choice of gauge*. Let ω be the associated connection 1 form; $\nabla e_i = \omega_i^j e_j$. We define

$$\Theta := \Theta(\nabla) := \mathrm{d}g\, g^{-1} + g\omega g^{-1},$$
$$\Omega := \Omega(\nabla) := g(\mathrm{d}\omega - \omega \wedge \omega)g^{-1} = g(\pi^* R)g^{-1}.$$

These are Lie algebra valued forms on the principal bundle $\mathcal{P}$ which do not depend on the local frame field chosen. If $Q \in \mathcal{Q}_\nu$, then we have $Q(\Omega) = \pi^* Q(\nabla)$. We set $\Omega(t) = t\,\mathrm{d}\Theta - t^2\Theta \wedge \Theta = t\Omega + (t - t^2)\Theta \wedge \Theta$ and define

$$\mathcal{T}Q(\nabla) := \nu \int_0^1 Q(\Theta, \Omega(t), \ldots, \Omega(t))\,\mathrm{d}t. \tag{4.2}$$

We refer to [10, Propositions 3.2, 3.7, and 3.8] for the proof of:

Theorem 4.1. *Let $Q \in \mathcal{Q}_\nu$ and let $\tilde{Q} \in \mathcal{Q}_\mu$.*
(1) *We have $\mathrm{d}\mathcal{T}Q(\nabla) = \pi^* Q(\nabla)$.*
(2) *We have $\mathcal{T}(Q\tilde{Q})(\nabla) = \mathcal{T}Q(\nabla) \wedge \pi^*\tilde{Q}(\nabla)$+exact $= \pi^* Q(\nabla) \wedge \mathcal{T}\tilde{Q}(\nabla)$+exact.*
(3) *Let ∇_ϱ be a smooth 1-parameter family of connections. Let $A := \partial_\varrho \nabla_\varrho|_{\varrho=0}$. Then*

$$\partial_\varrho \mathcal{T}Q(\nabla_\varrho)|_{\varrho=0} = \nu Q(A, \Omega_0, \ldots, \Omega_0) + exact.$$

Suppose M is parallelizable. Let e be a global frame for the principal frame bundle $\mathcal{P}$. Let ${}^e\nabla e = 0$ define the connection ${}^e\nabla$. Let $\omega_e = \nabla e$ and let

$$\mathcal{R}_t := t\,\mathrm{d}\omega_e - t^2\omega_e \wedge \omega_e = t\mathcal{R} + (t - t^2)\omega_e \wedge \omega_e.$$

We use Eqs. (4.1) and (4.2) to see that

$$e^*\mathcal{T}Q(\nabla) = \int_0^1 Q(\omega_e, \mathcal{R}_t, \ldots, \mathcal{R}_t) = TQ(\nabla, {}^e\nabla). \tag{4.3}$$

We note that $\mathcal{R}_t$ is the curvature of the connection $t{}^e\nabla + (1 - t)\nabla$. Fix $g \in \mathrm{GL}(m; \mathbb{R})$. Since Q is GL invariant, we have

$$e^*\mathcal{T}Q(\nabla) = (ge)^*\mathcal{T}Q(\nabla). \tag{4.4}$$

Let $Q \in \mathcal{Q}_\nu$. Suppose that $Q(\nabla) = 0$. Then $e^*\mathcal{T}Q(\nabla)$ is a closed form on M of degree $2\nu - 1$ and $[e^*\mathcal{T}Q(\nabla)]$ in $H^{2\nu-1}(M; \mathbb{C})$ is independent of the homotopy class of e. We say that Q is integral if Q is the image of an integral class in the classifying space; see [10, Section 3] for details; the Pontrjagin polynomials are integral.

Theorem 4.2. *Let $Q \in \mathcal{Q}_\nu$. Assume that M is parallelizable and that $Q(\nabla) = 0$.*
(1) *If Q is integral, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.*
(2) *If ν is odd, then $[e^*\mathcal{T}Q(\nabla)]$ is independent of e in $H^{2\nu-1}(M; \mathbb{C})$.*

Proof. We refer to [10, Theorem 3.16] for the proof of the first assertion; the second assertion follows from the first since the real cohomology of the classifying space vanishes in dimensions k which are not congruent to 0 mod 4. □

5. Affine invariance of the secondary characteristic classes

Let $(\nabla, h, {}^*\nabla)$ be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Since y is transverse to the hypersurface, the map $\psi_{x,X,y} := x_* \oplus y$ defines an isomorphism between $TM \oplus 1$ and the trivial bundle $M \times V$. Choose a basis $\{e_1, \dots, e_{m+1}\}$ for the underlying vector space V. We then have that $e_{x,X,y} := \psi^{-1}_{x,X,y}e$ is a stable parallelization of M; this depends, of course, on the immersion x and upon the relative normalization (X, y). Let $\mathcal{M} := M \times \mathbb{R}$; $T\mathcal{M} = TM \oplus 1$ so $e_{x,X,y}$ gives a parallelization of $\mathcal{M}$. We have increased the structure group from $\mathrm{GL}(m, \mathbb{R})$ to $\mathrm{GL}(m+1, \mathbb{R})$ in order to use Theorem 4.2.

Let $Q \in \mathcal{Q}_\nu$. Let ${}_M\nabla$ be a torsion free connection on M with $Q({}_M\nabla) = 0$. We extend ${}_M\nabla$ to a torsion free connection ${}_\mathcal{M}\nabla$ on $\mathcal{M}$ by defining ${}_\mathcal{M}\nabla\partial_t = 0$. We may express Q in terms of traces since traces generate the characteristic ring. Thus Q extends to an invariant polynomial on $\mathfrak{gl}(m+1, \mathbb{R})$. Since ${}_\mathcal{M}\mathcal{R} = {}_M\mathcal{R}\oplus 0$, we have $Q({}_\mathcal{M}\nabla) = Q({}_M\nabla) = 0$. Thus $e^*_{x,X,y}\mathcal{T}Q({}_\mathcal{M}\nabla)$ is a well-defined closed differential form on $\mathcal{M}$ which is independent of the auxilary parameter t and which therefore restricts to a well-defined closed differential form on M. The space of relative normalizations is path connected once an orientation is chosen; thus $[e^*_{x,X,y}\mathcal{T}Q({}_\mathcal{M}\nabla)]$ is independent of the relative normalization; it is also independent of the particular basis for V chosen. We denote this cohomology class by

$$[\mathcal{T}_x Q({}_M\nabla)] \in H^{2\nu-1}(M; \mathbb{C}).$$

By Theorem 3.2, $Q(\nabla) = 0$, $Q({}^h\nabla) = 0$, and $Q({}^*\nabla) = 0$. Thus we can apply this construction to the three natural connections associated with the relative normalization. We say that Q is *decomposible* if $Q = \sum_i Q_{i,1}Q_{i,2}$ where the $Q_{i,j}$ are non-trivial invariant polynomials which are homogeneous of positive degree. We begin our study with the following lemma.

Lemma 5.1. *Let $(\nabla, h, {}^*\nabla)$ be the conjugate triple defined by a relative normalization (x, X, y) of an affine embedding of an orientable manifold M. Let $Q \in \mathcal{Q}_\nu$.*
(1) *If Q is decomposible, then $[\mathcal{T}_x Q(\nabla)] = 0$, $[\mathcal{T}_x Q({}^*\nabla)] = 0$, and $[\mathcal{T}_x Q({}^h\nabla)] = 0$ in $H^{2\nu-1}(M; \mathbb{C})$.*

(2) *The classes* $[\mathcal{T}_x Q(\nabla)]$, $[\mathcal{T}_x Q(^*\nabla)]$, *and* $[\mathcal{T}_x Q(^h\nabla)]$ *in* $H^{2\nu-1}(M;\mathbb{C})$ *are affine invariants; these cohomology classes are independent of the relative normalization chosen.*

Proof. Suppose Q is decomposible. Let $\bar\nabla$ be one of the three connections in question. We use Theorem 4.1(2) to see $\mathcal{T}_x Q(\bar\nabla) = \sum_i \mathcal{T}_x Q_{i,1}(\bar\nabla)\wedge Q_{i,2}(\bar\nabla)$+exact. We apply Theorem 3.2 to see $Q_{i,2}(\bar\nabla)=0$. This proves the first assertion.

Without loss of generality, we may assume $Q(\Omega)=\mathrm{Tr}(\Omega^{\mu+1})$ since such traces generate the characteristic ring. Let $\beta(\varrho) := e^{\varrho\alpha}$ be a 1-parameter family of the gauge group $\mathcal{G}$. Let $\bar\nabla(\varrho)$ be the associated 1-parameter family of connections where we use the transformation laws described in Section 3. Let $\bar A = \partial_\varrho \bar\nabla(\varrho)|_{\varrho=0}$. We define

$$\mathcal{E} := \bar A_{i_0 j_{2\mu}}{}^{j_1} \bar R_{i_1 i_2 j_1}{}^{j_2}\cdots \bar R_{i_{2\mu-1} i_{2\mu} j_{2\mu-1}}{}^{j_{2\mu}} e^{i_0}\wedge e^{i_1}\wedge\cdots\wedge e^{i_{2\mu}}.$$

If we can show $\mathcal{E}=0$, the desired result will follow from Theorem 4.1(3). Since $\mathcal{E}$ is independent of the particular frame chosen, we can compute in a local frame field which is orthonormal with respect to the metric h.

Suppose first $\bar\nabla = {}^*\nabla$. By Eq. (3.3), ${}^*A_{ijk} = (e_i\alpha)\delta_{jk} + (e_j\alpha)\delta_{ik}$. Thus

$$\mathcal{E} = \mathrm{d}\alpha\wedge \mathrm{Tr}((^*R)^\mu) + e_{j_{2\mu}}(\alpha)\delta_{i_0 j_1}{}^*R_{i_1 i_2 j_1 j_2}\cdots e^{i_0}\wedge e^{i_1}\wedge e^{i_2}\cdots$$

The first term vanishes by Theorem 3.2. Since ${}^*\nabla$ is torsion free, ${}^*R$ satisfies the Bianchi identity so $R_{i_1 i_2 i_0 j_1} e^{i_0}\wedge e^{i_1}\wedge e^{i_2} = 0$ and the second term vanishes as well.

Suppose next that $\bar\nabla = \nabla$. By Eq. (3.3), $A_{ijk} = -e_k(\alpha)\delta_{ij}$. Thus

$$\mathcal{E} = -e_{j_1}(\alpha)\delta_{i_0 j_{2\mu}} R_{i_1 i_2 j_1 j_2}\cdots R_{i_{2\mu-1} i_{2\mu} j_{2\mu-1} j_{2\mu}} e^{i_0}\wedge e^{i_2}\wedge e^{i_3}\cdots e^{i_{2\mu}}.$$

We use Eq. (3.4) to see $R_{abcd} = \delta_{bc}S_{ad} - \delta_{ad}S_{bc}$. Since S is symmetric and since R satisfies the Bianchi identities, we show $\mathcal{E}=0$ by computing

$$R_{i_{2\mu-1} i_{2\mu} j_{2\mu-1} i_0} e^{i_0}\wedge e^{i_{2\mu-1}}\wedge e^{i_{2\mu}} = -R_{i_{2\mu-1} i_{2\mu} i_0 j_{2\mu-1}} e^{i_0}\wedge e^{i_{2\mu-1}}\wedge e^{i_{2\mu}} = 0.$$

Finally, we consider the connection ${}^h\nabla$. Relative to a coordinate frame, we have

$${}^h\Gamma_{ij}{}^k = \tfrac12 h^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}).$$

Consequently, relative to an orthonormal frame, we have

$${}^hA_{ijk} = \tfrac12\{\delta_{ik}e_j(\alpha) + \delta_{jk}e_i(\alpha) - \delta_{ij}e_k(\alpha)\}.$$

We decompose $\mathcal{E} = \frac12(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)$ where

$$\begin{aligned}
\mathcal{E}_1 &= \alpha_{;j_{2\mu}}\delta_{i_0 j_1}{}^hR_{i_1 i_2 j_1 j_2}\cdots {}^hR_{i_{2\mu-1} i_{2\mu} j_{2\mu-1}}{}^{j_{2\mu}} e^{i_0}\wedge e^{i_1}\wedge\cdots\wedge e^{i_{2\mu}},\\
\mathcal{E}_2 &= \alpha_{;i_0}\delta_{j_{2\mu} j_1}{}^hR_{i_1 i_2 j_1 j_2}\cdots {}^hR_{i_{2\mu-1} i_{2\mu} j_{2\mu-1}}{}^{j_{2\mu}} e^{i_0}\wedge e^{i_1}\wedge\cdots\wedge e^{i_{2\mu}},\\
\mathcal{E}_3 &= -\alpha_{;j_1}\delta_{i_0 j_{2\mu}}{}^hR_{i_1 i_2 j_1 j_2}\cdots {}^hR_{i_{2\mu-1} i_{2\mu} j_{2\mu-1}}{}^{j_{2\mu}} e^{i_0}\wedge e^{i_1}\wedge\cdots\wedge e^{i_{2\mu}}.
\end{aligned}$$

We use the Bianchi identity to see $\mathcal{E}_1 = 0$. Since $\mathcal{E}_2 = \mathrm{d}\alpha\wedge\mathrm{Tr}((^hR)^{\mu-1}) = 0$, $\mathcal{E}_2=0$ by Theorem 3.2. Since $\mathcal{E}_3$ involves ${}^hR_{i_{2\mu-1} i_{2\mu} j_{2\mu-1} i_0} e^{i_0}\wedge e^{i_{2\mu-1}}\wedge e^{i_{2\mu}}$, $\mathcal{E}_3 = 0$ because ${}^hR_{abcd} = -{}^hR_{abdc}$ and because ${}^hR$ satisfies the Bianchi identity. □

We use Lemma 5.1 to prove the following result which is one of the main results of this paper:

Theorem 5.2. *Let* $(\nabla, h, {}^*\nabla)$ *be the conjugate triple defined by a relative normalization* (x, X, y) *of an affine embedding of an orientable manifold* M. *Let* $Q \in \mathcal{Q}_\nu$.
(1) *We have* $[\mathcal{T}_x Q(\nabla)] = 0$ *in* $H^{2\nu-1}(M; \mathbb{C})$.
(2) *If* Q *is integral and if* ν *is even, then* $[\mathcal{T}_x Q({}^*\nabla) = 0]$ *in* $H^{2\nu-1}(M; \mathbb{C}/\mathbb{Z})$.
(3) *If* ν *is odd, then* $[\mathcal{T}_x Q({}^*\nabla) = 0]$ *in* $H^{2\nu-1}(M; \mathbb{C})$.
(4) *If* ν *is even, then* $[\mathcal{T}_x Q({}^h\nabla) = 0]$ *in* $H^{2\nu-1}(M; \mathbb{C})$.
(5) *If* ν *is odd and if* h *is definite, then* $[\mathcal{T}_x Q({}^h\nabla)] = 0$ *in* $H^{2\nu-1}(M; \mathbb{C})$.

Proof. As in the proof of Lemma 5.1, we may suppose without loss of generality that $Q(A) = \mathrm{Tr}(A^\nu)$. By Lemma 5.1, the cohomology class of $[\mathcal{T}_x Q(\cdot)]$ is independent of the relative normalization in $H^{2\nu-1}(M; \mathbb{C})$. Thus we may choose a convenient relative normalization to prove Theorem 5.2. We choose an inner product and an origin to identify the affine space with $\mathbb{R}^{m+1}$. Let N be the normal vector to the embedding. We use the Euclidean inner product to identify V with V^*. We let $X = N$ and $y = N$. We show that (x, X, y) is a relative normalization by checking the equations of structure:

$$\langle X, \mathrm{d}x(u)\rangle = (N, \partial_u x) = 0, \quad \langle X, y\rangle = (N, N) = 1,$$
$$\langle \mathrm{d}X(u), y\rangle = (\partial_u N, N) = \tfrac{1}{2}\partial_u(N, N) = 0.$$

Let v and w be tangent vector fields on M. Let

$$g(v, w) := (v\boldsymbol{x}, w\boldsymbol{x}),$$
$$h(u, v) := (vw\boldsymbol{x}, \boldsymbol{N}) = -(v\boldsymbol{x}, w\boldsymbol{N}),$$
$$S(u, v) := (u\boldsymbol{N}, v\boldsymbol{N}) = -(uv\boldsymbol{N}, \boldsymbol{N})$$

be the first, second, and third fundamental forms of the immersion, respectively. Fix a point P in M. We shift the origin and rotate the coordinate axes if need be to assume that $x(P) = 0$ and that $N(P) = (0, \ldots, 0, 1)$. We write x as a graph over the coordinate hyperplane near P to express $x(u) = (u_1, \ldots, u_m, f)$ for some smooth function $f(u)$ defined near $\boldsymbol{u} = 0$. Let Roman indices range from 1 to m and let Greek indices range from 1 to $m + 1$. Let $\partial_i = \partial/\partial u_i$. We compute

$$\partial_i\boldsymbol{x} = (0, \ldots, 0, 1, 0, \ldots, 0, \partial_i f),$$
$$g_{ij} = \delta_{ij} + (\partial_i f)(\partial_j f),$$
$$N = (-\partial_1 f, \ldots, -\partial_m f, 1)\left(1 + \sum_i (\partial_i f)^2\right)^{-1}.$$

Since $\boldsymbol{N}(0) = (0, \ldots, 0, 1)$, we have $\mathrm{d}f(P) = 0$ and thus we have

$$N = (-\partial_1 f, \ldots, -\partial_m f, 1) + \mathrm{O}(u^2), \quad \text{and} \quad h_{ij} = \partial_i\partial_j f + \mathrm{O}(u^2).$$

Since $g_{ij} = \delta_{ij} + \mathrm{O}(u^2)$, the Christoffel symbols ${}^g\nabla_{\partial_i}\partial_j$ of the Levi-Civita connection defined by the metric g vanish at P. We use Eq. (3.2) to see that $\nabla = {}^g\nabla$. We use Eq. (4.4) to see that we can renormalize the basis e for $\mathbb{R}^{m+1}$ to assume that e_{m+1} is the normal at P. We use ψ to identify $T\mathcal{M}$ with $\mathcal{M} \times \mathbb{R}^{m+1}$. We compute

$$\begin{aligned} e_i &= (0,\dots,1,\dots,0) + \partial_i f(0,\dots,0,1) - \partial_i f(0,\dots,0,1) \\ &= \partial_i x - (\partial_i f)N + \mathrm{O}(u^2), \\ e_{m+1} &= (-\partial_1 f,\dots,-\partial_m f,1) + (\partial_1 f,\dots,\partial_m f,0) = N + \sum_i \partial_i(f)\partial_i x + \mathrm{O}(u^2). \end{aligned}$$

Let $\mathcal{A} := {}^g_{\mathcal{M}}\omega_e(P)$ and let $\mathcal{R} := {}^g_{\mathcal{M}}R(P)$. We have

$$\mathcal{A}_j{}^{m+1} = -\mathcal{A}_{m+1}{}^j = -h_{ij}e^i, \quad \text{and} \quad \mathcal{A}_\alpha{}^\beta = 0, \quad \text{otherwise.}$$

We use Eq. (4.3) to see that we can express $\mathcal{T}_x Q({}^g\nabla)$ as the sum of traces of products of $\mathcal{A}$ and $\mathcal{R}$ with an odd number of $\mathcal{A}$ factors. Since $\mathcal{R}_i{}^{m+1} = 0$ and $\mathcal{R}_{m+1}{}^i = 0$, these traces vanish; this proves the first assertion.

The connection ${}^*\nabla$ is the Levi-Civita connection of the embedding X. Thus $[\mathcal{T}_X Q({}^*\nabla)] = 0$ in $H^{2\nu-1}(M;\mathbb{C})$ by Theorem 5.2(1). Since $\mathcal{T}_X$ and $\mathcal{T}_x$ reflect the pull-back by two different stable parallelizations of M, we use Theorem 4.2 to derive Theorem 5.2(2) and Theorem 5.2(3) from Theorem 5.2(1).

In the proof of the final assertion, we first suppose ν is even. We clear the previous notation. Let $\mathcal{A} := {}^g_{\mathcal{M}}\omega_e(P)$, ${}^h\mathcal{R} = {}^h_{\mathcal{M}}\mathcal{R}_e(P)$ and ${}^h\mathcal{C}_j{}^k = C(P)_{ij}{}^k e^i$. We then have ${}^h_{\mathcal{M}}\omega_e(P) = \mathcal{C} + \mathcal{A}$. It is clear that $\mathcal{T}_x Q({}^h\nabla)$ is the trace of non-commutative monomials in $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{A}$. We note that

$$\begin{aligned} &\mathcal{A}_j{}^{m+1} = -\mathcal{A}_{m+1}{}^j = -h_{ij}e^i, \quad \mathcal{A}_\alpha{}^\beta = 0, \quad \text{otherwise,} \\ &{}^h\mathcal{R}_{m+1}{}^i = {}^h\mathcal{R}_i{}^{m+1} = {}^h\mathcal{R}_{m+1}{}^{m+1} = 0, \\ &\mathcal{C}_{m+1}{}^i = \mathcal{C}_i{}^{m+1} = \mathcal{C}_{m+1}{}^{m+1} = 0. \end{aligned}$$

Thus if $\mathcal{A}$ appears, it must touch itself, so $\mathcal{T}_x Q({}^h\nabla)$ is the trace of non-commutative monomials in $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{A}^2$. We note that

$$(\mathcal{A}^2)_k{}^l = -h_{ik}h_{jl}e^i \wedge e^j, \qquad (\mathcal{A}^2)_{m+1}^{m+1} = -h_{ij}h_{jk}e^i \wedge e^k = 0,$$

and $(\mathcal{A}^2)_\alpha{}^\beta = 0$ otherwise. We use the facts that C_{ijk} is symmetric and that ${}^hR$ satisfies the Bianchi identity to see

$$\begin{aligned} \mathcal{C}_{j_1}{}^{j_2}(\mathcal{A}^2)_{j_2}{}^{j_3} &= C_{i_0 j_1}{}^{j_2} h_{i_1 j_2} h_{i_2 j_3} e^{i_0} \wedge e^{i_1} \wedge e^{i_2} \\ &= C_{i_0 j_2 i_1} h_{i_2 j_3} e^{i_0} \wedge e^{i_1} \wedge e^{i_2} = 0, \\ {}^h\mathcal{R}_{i_0 i_1 j_1}{}^{j_2}(\mathcal{A}^2)_{j_2}{}^{j_3} &= {}^h\mathcal{R}_{i_0 i_1 j_1}{}^{j_2} h_{i_2 j_2} h_{i_3 j_3} e^{i_0} \wedge e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \\ &= {}^h\mathcal{R}_{i_0 i_1 j_1 i_2} h_{i_3 j_3} e^{i_0} \wedge e^{i_1} \wedge e^{i_2} \wedge e^{i_3} = 0. \end{aligned}$$

This shows that we may ignore the role of $\mathcal{A}$ in our computations. We change our point of view and compute with respect to a frame which is orthonormal with respect to h at P. We use Eq. (3.4) to express ${}^h\mathcal{R}$ in terms of C and S. The argument given to prove Theorem 3.2 shows that the terms involving S yield 0. Thus we may replace ${}^h\mathcal{R}$ by $\mathcal{C}^2$ and express

$$\mathcal{T}_x Q({}^h\nabla) = \kappa_\nu \operatorname{Tr}(\mathcal{C}^{2\nu-1}) = \kappa_\nu C_{i_1 j_1 j_2} C_{i_2 j_2 j_3} \cdots C_{i_{2\nu-1} j_{2\nu-1} j_1} e^{i_1} \wedge \dots,$$

where κ_ν is a certain univeral constant. When we take the transpose, we introduce a factor of $(-1)^{\nu+1}$ since we are working with differential forms:

$$e^{i_1} \wedge \cdots \wedge e^{i_{2\nu-1}} = (-1)^{\nu+1} e^{i_{2\nu-1}} \wedge \cdots \wedge e^{i_1},$$

so

$$\operatorname{Tr}(\mathcal{C}^{2\nu-1}) = (-1)^{\nu+1} \operatorname{Tr}((\mathcal{C}^{2\nu-1})^t) = (-1)^{\nu_1} \operatorname{Tr}((\mathcal{C}^t)^{2\nu-1}).$$

Since C is symmetric, $\operatorname{Tr}((\mathcal{C}^t)^{2\nu-1}) = \operatorname{Tr}(\mathcal{C}^{2\nu-1})$ and the desired vanishing theorem now follows if ν is even.

Suppose that ν is odd. Since the metric h is definite, we may apply the Gram–Schmidt process to construct a parallelization $\bar{e}$ of $\mathcal{M}$ which is orthonormal with respect to the metric $h \oplus \mathrm{d}t^2$ and which is homotopic to the original frame e. We clear the previous notation. Let ${}^h_{\mathcal{M}}\nabla \bar{e}_i = \omega_{ij}\bar{e}_j$. Since ${}^h_{\mathcal{M}}\nabla$ is Riemannian, we have $\omega_{ij} = -\omega_{ji}$ and $\mathcal{R}_{ij} = -\mathcal{R}_{ji}$. We define $\mathcal{R}(t) := t\mathcal{R} + (t - t^2)\omega \wedge \omega$. We then have $\mathcal{R}(t)_{ij} = -\mathcal{R}(t)_{ji}$. We use this skew-symmetry to compute

$$\begin{aligned}
\bar{e}^* \mathcal{T} Q({}^h_{\mathcal{M}}\nabla) &= \nu \int_0^1 \omega_{i_1 i_2} \mathcal{R}(t)_{i_2 i_3} \cdots \mathcal{R}(t)_{i_\nu i_1}\, \mathrm{d}t \\
&= (-1)^\nu \nu \int_0^1 \omega_{i_2 i_1} \mathcal{R}(t)_{i_1 i_\nu} \cdots \mathcal{R}(t)_{i_3 i_2}\, \mathrm{d}t \\
&= (-1)^\nu \bar{e}^* \mathcal{T} Q({}^h_{\mathcal{M}}\nabla).
\end{aligned}$$

If ν is odd, this implies $2\bar{e}^* \mathcal{T} Q({}^h_{\mathcal{M}}\nabla) = 0$. □

6. Affine geometry in three dimensions

Let M be a compact orientable three-dimensional manifold. Then M is parallelizable; we choose a global frame f for TM. If $Q \in \mathcal{Q}_2$, then $Q = cP_1 +$ decomposible, so we need only study $[\mathcal{T}_x P_1]$, where P_1 is the first Pontrjagin form. Note that P_1 is a real integral differential form. We define

$$\Phi(\nabla) = \int_M f^* \mathcal{T} P_1(\nabla) \in \mathbb{R}/\mathbb{Z},$$

by Theorem 4.2, this is independent of the particular parallelization f which is chosen. We use Theorem 5.2 to see:

Theorem 6.1. *Let (M, g_0) be a three-dimensional Riemannian manifold:*
(1) *If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the first fundamental form of x, then $\Phi_e({}^g\nabla) = 0$ in $\mathbb{R}/\mathbb{Z}$.*
(2) *If there exists an immersion $x : M \to \mathbb{R}^4$ so that g_0 is conformally equivalent to the second fundamental form of x, then $\Phi_e({}^g\nabla) = 0$ in $\mathbb{R}/\mathbb{Z}$.*

We note that assertion (1) was first proved by Chern and Simons [10, Theorem 6.4]. They also showed that given an arbitrary real number r, there exists a left invariant metric on S^3 such that $\Phi_e(M, g_0) = r$; Theorem 6.1 shows that these metrics cannot be realized either as the first or the second fundamental form of an embedding in $\mathbb{R}^4$ if $r \notin \mathbb{Z}$.

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References

[1] M.F. Atiyah, R. Bott, V.K. Patodi, On the heat equation and the index theorem, Invent. Math. 13 (1973) 279–330.
[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry I, II, III, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69; 78 (1975) 405–432; 79 (1976) 71–99.
[3] O. Backofen, Composite link polynomials from super Chern–Simons theory, J. Geom. Phys. 20 (1996) 19–30.
[4] C. Baikoussis, T. Koufogiorgos, Pairs of metrics and Codazzi tensors on manifolds, An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Noua, Mat. 36 (1990) 403–409.
[5] N. Bokan, P. Gilkey, U. Simon, Applications of spectral geometry to affine and projective geometry, Contrib. Algebra and Geom. 35 (1994) 283–314.
[6] N. Bokan, P. Gilkey, U. Simon, Geometry of differential operators on Weyl manifolds, Proc. A Royal Soc. London 453 (1997) 2527–2536.
[7] D. Chae, N. Kim, Topological multivortex solutions of the self-dual Maxwell–Chern–Simons–Higgs system, J. Differential Equations 134 (1995) 154–182.
[8] S. Chern, On the curvatura integra in a Riemannian manifold, Ann. of Math. 46 (1945) 674–684.
[9] S. Chern, J. Simons, Some cohomology classes in principal fiber bundles and their applications to Riemannian geometry, Proc. Nat. Acad. Sci. USA 68 (1971) 791–794.
[10] S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. of Math. 99 (1974) 48–69.
[11] M.A. Cheshkova, Connections associated with a Codazzi tensor field, Tr. Geom. Semin. 22 (1994) 89–90.
[12] T. Eguchi, P.B. Gilkey, A.J. Hanson, Gravitation, gauge theory, and differential geometry, Phys. Rep. 66 (1980) 213–393.
[13] B. Fine, P. Kirk, E. Klassen, A local analytic splitting of the holonomy map on flat connections, Math. Ann. 299 (1994) 171–189.
[14] G.B. Folland, Weyl manifolds, J. Differential Geom. 4 (1970) 145–153.
[15] D. Freed, Classical Chern Simons Theory I, Adv. Math. 113 (1995) 237–303.
[16] P.B. Gilkey, On the index of geometrical operators on Riemannian manifolds with boundary, Adv. in Math. 102 (1993) 129–183.
[17] P.B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem, 2nd ed., CRC Press, Baca Raton, FL, 1994, ISBN 0-8493-7874-4.

[18] K. Haller, L. Lombridas, Maxwell–Chern–Simons theory in covariant and Coulomb gauges, Ann. Phys. 246 (1996) 1–48.
[19] T. Higa, Weyl manifolds and Einstein–Weyl manifolds, Comm. Math. Univ. Sancti Pauli 42 (1993) 143–160.
[20] N.J. Hitchin, Complex manifolds and Einstein's equation, Springer Lecture Notes 970 (1982) 73–99.
[21] P. Horava, Chern–Simons gauge theory on orbifolds: Open strings from three dimensions, J. Geom. Phys. 21 (1996) 1–33.
[22] P. Kirk, E. Klassen, Chern–Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of T^2, Comm. Math. Phys. 153 (1993) 521–557.
[23] M. Ouyang, A note on the Chern–Simons invariant of hyperbolic 3-manifolds, PAMS 125 (1997) 1845–1851.
[24] O. Pashaev, The Lax pair by dimensional reduction of Chern–Simons gauge theory, J. Math. Phys. 37 (1996) 4368–4387.
[25] H. Pedersen, A. Swann, Riemannian submersions, four manifolds, and Einstein–Weyl geometry, Proc. London Math. Soc. 66 (1991) 381–399.
[26] U. Pinkall, A. Schwenk-Schellschmidt, U. Simon, Geometric methods for solving Codazzi and Monge-Amère equations, Math. Ann. 298 (1994) 89–100.
[27] G. Saban, Some elementary considerations on Codazzi pairs, Rend. Sem. Mat. Fis. Milano 54 (1984) 225–237.
[28] Y. Shen, Harmonic Gauss maps and Codazzi tensors for affine hypersurfaces, Arch. Math. 55 (1990) 298–305.
[29] U. Simon, A. Schwenk-Schellschmidt, H. Viesel, Introduction to the affine differential geometry of hypersurfaces, Lecture Notes, Science University, Tokyo, 1991, ISBN 3798315299.
[30] U. Simon, Transformation techniques for PDEs on projectively flat manifolds, Result Math. 27 (1995) 160–187.
[31] H. Weyl, Space–time Matter, Dover New York, 1922.
[32] H. Weyl, The Classical Groups, Princeton University Press, Princeton, NJ, 1946.
[33] Y .Yang, The relativistic non-Abelian Chern–Simons equations, Comm. Math. Phys. 186 (1997) 199–218.